Mathematics

GE - 4

Semester - <u>IV</u>

Numerical Methods

Relation between the operators :

1. $\Delta = E - 1$
2. $\nabla = 1 - E^{-1}$
3. $E = e^{hD}$
4. $D = \frac{1}{h}\log(1+\Delta)$, where $D = \frac{d}{dx}$

Proof : (1) $\Delta y(x) = y(x+h) - y(x)$

$= Ey(x) - y(x) = (E-1)\, y(x)$

$\Rightarrow \boxed{\Delta = E - 1}$ proved.

(2) $\nabla y(x) = y(x) - y(x-h)$

$= y(x) - E^{-1} y(x)$

$= (1 - E^{-1})\, y(x)$

$\Rightarrow \nabla = \boxed{1 - E^{-1}}$. proved.

(3) $E f(x) = f(x+h)$

$= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots$

(Using Taylor's Thm.)

$= f(x) + h D f(x) + \frac{h^2}{2!} D^2 f(x) + \cdots$

$= \left(1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \cdots\right) f(x)$

$= e^{hD} f(x)$ $\left[\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$

$\Rightarrow \boxed{E = e^{hD}}$ proved.

(4) From (1),

$$E = 1 + \Delta$$

From (5),

$$E = e^{hD}$$

$$\Rightarrow e^{hD} = 1 + \Delta$$

Taking log on both sides,

$$hD = \log(1+\Delta)$$

$$\Rightarrow \boxed{D = \frac{1}{h}\log(1+\Delta)}$$ proved

It may also be written as,

$$D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \cdots\right].$$

Numerical Differentiation

Numerical differentiation is the process of calculating the derivative of a function at some particular value of the independent variable by means of a set of given values of that function.

Consider the function $y = f(x)$, which is tabulated as —

x	$f(x)$
x_0	y_0
x_0+h	y_1
x_0+2h	y_2
$\vdots$	$\vdots$
x_0+nh	y_n

(1) Derivatives using forward difference formula

We know that

$$1+\Delta = e^{hD}$$

$$\Rightarrow \log(e^{hD}) = \log(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots$$

$$\Rightarrow hD = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots \longrightarrow (1)$$

$$\Rightarrow D = \frac{1}{h}\left(\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots\right)$$

$$\Rightarrow Df(x) = \frac{1}{h}\left(\Delta f(x) - \frac{1}{2}\Delta^2 f(x) + \frac{1}{3}\Delta^3 f(x) - \frac{1}{4}\Delta^4 f(x) + \cdots\right)$$

At $x = x_0$,

$$Df(x_0) = \frac{1}{h}\left(\Delta f(x_0) - \frac{1}{2}\Delta^2 f(x_0) + \frac{1}{3}\Delta^3 f(x_0) - \frac{1}{4}\Delta^4 f(x_0) + \cdots\right)$$

i.e.,

$$f'(x_0) = \frac{1}{h}\left[\Delta f(x_0) - \frac{1}{2}\Delta^2 f(x_0) + \frac{1}{3}\Delta^3 f(x_0) - \frac{1}{4}\Delta^4 f(x_0) + \cdots\right]$$

C.T.M.

This is the formula to be used to evaluate first order derivative at a point in the begining of the given data range.

Now, squaring both sides of eqn. (1),

$$(hD)^2 = \left(\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots\right)^2$$

$$= \Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \frac{137}{180}\Delta^6 + \cdots$$

$$\Rightarrow D^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \frac{137}{180}\Delta^6 + \cdots\right]$$

$$\Rightarrow D^2 f(x) = \frac{1}{h^2}\left[\Delta^2 f(x) - \Delta^3 f(x) + \frac{11}{12}\Delta^4 f(x) - \frac{5}{6}\Delta^5 f(x) + \frac{137}{180}\Delta^6 f(x) + \cdots\right]$$

$$\Rightarrow f''(x) = \quad \downarrow$$

At $x = x_0$,

$$f''(x_0) = \frac{1}{h^2}\left[\Delta^2 f(x_0) - \Delta^3 f(x_0) + \frac{11}{12}\Delta^4 f(x_0) - \frac{5}{6}\Delta^5 f(x_0) + \frac{137}{180}\Delta^6 f(x_0) + \cdots\right]$$

C.T.M.

which is the formula to be used to evaluate second order derivative at a point in the begining of the given data range.

On the same lines, we can find the formula for derivative of third order at a point in the begining of the tabulated values, which is given as —

$$f'''(x_0) = \frac{1}{h^3}\left[\Delta^3 f(x_0) - \frac{3}{2}\Delta^4 f(x_0) + \cdots\right]$$

(Prove it yourself). $\boxed{\text{Exercise}}$.

Note : Above three formulas can also be obtained using Newton's forward difference formula. For this, first write down the Newton's forward formula. Then differentiate it w.r.t. h to obtain $\frac{dy}{dh}$.

Then use chain rule to obtain $\frac{dy}{dx}$:—

$$\frac{dy}{dx} = \frac{dy}{dh} \times \frac{dh}{dx}$$

where $x = x_0 + ih$, $i = 0, 1, 2, \ldots$

So, $\frac{dh}{dx} = \frac{1}{i}$.

Finally Put $x = x_0$ and $h = 0$.

Try yourself.

For second order derivative,

$$\frac{d^2y}{dx^2} = \frac{d}{dh}\left(\frac{dy}{dx}\right)\frac{dh}{dx}$$

Finally Put $x = x_0$ and $h = 0$.

Try yourself.

Note :- Newton's forward difference formula is applied when values of x are equidistant (equispaced) and the value of $\frac{dy}{dx}$ is required at a point in the starting of table.

Derivatives using backward difference formula :

We have

$$\nabla = 1 - E^{-1}$$

$$\Rightarrow E^{-1} = 1 - \nabla$$

$$\Rightarrow E = (1-\nabla)^{-1}$$

$$\Rightarrow e^{hD} = (1-\nabla)^{-1}$$

Taking log on both sides,

$$\log e^{hD} = \log(1-\nabla)^{-1}$$

$$\Rightarrow hD = \log(1-\nabla)^{-1} = (-1)\log(1-\nabla)$$

$$\Rightarrow hD = \nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \dots$$

$$\Rightarrow D = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \dots\right]$$

(7)

$$\Rightarrow D f(x) = \frac{1}{h}\left[\nabla f(x) + \frac{1}{2}\nabla^2 f(x) + \frac{1}{3}\nabla^3 f(x) + \frac{1}{4}\nabla^4 f(x) + \cdots\right]$$

$$\Rightarrow f'(x) = \frac{1}{h}\left[\nabla f(x) + \frac{1}{2}\nabla^2 f(x) + \frac{1}{3}\nabla^3 f(x) + \frac{1}{4}\nabla^4 f(x) + \cdots\right]$$

At $x = x_n$,

$$f'(x_n) = \frac{1}{h}\left[\nabla f(x_n) + \frac{1}{2}\nabla^2 f(x_n) + \frac{1}{3}\nabla^3 f(x_n) + \frac{1}{4}\nabla^4 f(x_n) + \cdots\right]$$

C.T.M.

This is the formula to be used to evaluate first order derivative at a point in the end of the given data points or tabulated values.

Now following the same procedure as in the case of forward difference formula, we get

$$f''(x_n) = \frac{1}{h^2}\left[\nabla^2 f(x_n) + \nabla^3 f(x_n) + \frac{11}{12}\nabla^4 f(x_n) + \frac{5}{6}\nabla^5 f(x_n) + \frac{137}{180}\nabla^6 f(x_n) + \cdots\right]$$

C.T.M

which is the formula to be used to evaluate second order derivative at a point in the ~~left~~ end of the given tabulated values.

On the same lines, we can find the formula for derivative of third order at a point in the end of the tabulated values, which is given as —

$$f'''(x_n) = \left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 f(x_n) + \frac{3}{2}\nabla^4 f(x_n) + \cdots\right]$$

C.T.M.

Prove it yourself. $\boxed{\text{Exercise}}$.

Note : Above three formulas can also be obtained using Newton's backward difference formula. For this, first write down the Newton's backward formula (interpolation). Then differentiate it w.r.t. h to obtain dy/dh.

Then use chain rule to obtain dy/dx as : —

$$\frac{dy}{dx} = \frac{dy}{dh} \times \frac{dh}{dx}$$

Since, $x = x_n + ih$, $i = 0, 1, 2, \ldots$

$$\therefore \frac{dh}{dx} = \frac{1}{i},$$

Then Put $x = x_n$ and $h = 0$.

Try yourself. $\boxed{\text{Exercise}}$.

For second order derivative,

$$\frac{d^2y}{dx^2} = \frac{d}{dh}\left(\frac{dy}{dx}\right)\frac{dh}{dx}.$$

Finally Put $x = x_n$ and $h = 0$.

Try yourself. $\boxed{\text{Exercise}}$.

Note : Newton's backward difference formula is applied when values of x are equispaced and the value of dy/dx is required at a point near the end of the table.

→ Now, if we require to evaluate dy/dx at a point in the middle of the table and the values of x are equispaced, then we use the formula for derivatives using Stirling's Central Difference formula and

Bessel's Central Difference formula.

→ When the values of x are not equispaced, then we use derivatives formula using Lagrange's formula or Newton's divided difference formula.

Example: The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds. Find the initial acc^n. using the data:

Time (t in seconds): 0 5 10 15 20

Velocity (v in m/sec): 0 2 13 68 227

Solution: Forward difference table for the given data is:

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
	Complete the table				

We will have

$$\Delta v_0 = 2, \quad \Delta^2 v_0 = 9, \quad \Delta^3 v_0 = 35, \quad \Delta^4 v_0 = 25$$

We need to evaluate initial acceleration at $t=0$ i.e., $\left(\dfrac{dv}{dt}\right)_{t=0}$.

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By Newton's forward formula for derivatives

$$\left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left(\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \cdots\right) \quad \text{(1)}$$

$$= \frac{1}{5}\left[2 - \frac{1}{2}(9) + \frac{1}{3}(35) - \frac{1}{4}(25)\right]$$

(All other forward differences required to put in formula (1) are zero).

$$\simeq 0.58$$

∴ Initial acceleration is $0.58\ m/sec^2$.

Practice Questions

① Find the value of $y'(0)$ and $y''(0)$ from the following table :

x	0	1	2	3	4	5
y	4	8	15	7	6	2

(Ans. → -27.9, 117.67)

② Find the value of $\cos(1.74)$ from the following table :

x	1.7	1.74	1.78	1.82	1.86
$\sin x$	0.9916	0.9857	0.9751	0.9691	0.9584

(Ans. → -0.175)

③ Given $\sin 0° = 0.000$, $\sin 10° = 0.1736$, $\sin 20° = 0.3420$, $\sin 30° = 0.5000$, $\sin 40° = 0.6428$

(a) Find the value of $\sin 23°$

(b) Find the numerical value of $\frac{dy}{dx}$ at $x = 10°$ for $y = \sin x$.

(c) Find the numerical value of $\frac{d^2y}{dx^2}$ at $x = 20°$ for $y = \sin x$.

(Ans → (a) 0.3907 (b) 0.9848 (c) 0.342)

Richardson Extrapolation :

This technique provides a method of improving the accuracy of a low-order approximation formula. i.e., this method is used to improve the numerical method's solution estimation.

Consider

$$x_k = x^* + Mh^n \longrightarrow (1)$$

where x_k is the kth estimate of true solution x^*, and Mh^n is the error term.

If we replace h by rh, then we get the another estimation for x^*.

i.e., $$x_{k+1} = x^* + Mr^n h^n \longrightarrow (2).$$

Eqn. (1) $\times r^n$ + (2) $\Rightarrow$

$$x_{k+1} - r^n x_k = (1-r^n)x^*$$

$$\Rightarrow \boxed{x^* = \frac{x_{k+1} - r^n x_k}{1-r^n}} \longrightarrow (3)$$

This x^* is also represented by x_R.

Equation (3) gives Richardson extrapolation estimate.

Using Richardson extrapolation, we get higher order formula from lower order formula, which improves the estimation.

Now, three-point central difference formula for first order derivative along with its error term is,

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(\theta) \longrightarrow (4)$$

Let us represent,

$$D(h) = \frac{f(x+h) - f(x-h)}{2h} \longrightarrow (5)$$

Then,

$$\underbrace{f'(x)}_{\text{Exact value}} = \underbrace{D(h)}_{\text{Approximated Value}} - \underbrace{\frac{h^2}{6} f'''(\theta)}_{\text{Error term}} \longrightarrow (6)$$

For better approximation for $f'(x)$, replace h by rh in equation (6),

$$f'(x) = D(rh) - \frac{r^2h^2}{6} f'''(\theta) \longrightarrow (7)$$

where

$$D(rh) = \frac{f(x+rh) - f(x-rh)}{2rh} \longrightarrow (8)$$

To eliminate the error term, apply eq^n. (7) — Eq^n. (6) $\times r^2$, we get

$$(1 - r^2) f'(x) = D(rh) - r^2 D(h)$$

$$\Rightarrow \boxed{f'(x) = \frac{D(rh) - r^2 D(h)}{1 - r^2}} \longrightarrow (9)$$

This is better estimation for $f'(x)$, as the error term having h^2 has been eliminated.

Case (i) :

For $r = 2$, eqn. (9) gives,

$$f'(x) = \frac{D(2h) - 4\,D(h)}{1-4}$$

$$\Rightarrow f'(x) = \frac{\left(\frac{f(x+2h) - f(x-2h)}{4h}\right) - 4\left(\frac{f(x+h) - f(x-h)}{2h}\right)}{-3}$$

$$\Rightarrow f'(x) = \frac{f(x+2h) - f(x-2h) - 8\,f(x+h) + 8\,f(x-h)}{-12h}$$

$$\Rightarrow \boxed{f'(x) = \frac{f(x-2h) - 8\,f(x-h) + 8\,f(x+h) - f(x+2h)}{12h}} \longrightarrow (10)$$

Equation (10) is five point central difference formula for $f'(x)$.

Case (ii) :

For $r = 0.5$, Eqn. (9) gives,

$$f'(x) = \frac{D\left(\frac{h}{2}\right) - \frac{1}{4} D(h)}{1 - \frac{1}{4}}$$

$$\Rightarrow f'(x) = \frac{\left(\dfrac{f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)}{2\cdot\frac{h}{2}}\right) - \frac{1}{4}\left(\dfrac{f(x+h)-f(x-h)}{2h}\right)}{3/4}$$

$$\Rightarrow f'(x) = \frac{8\,f\left(x+\frac{h}{2}\right) - 8f\left(x-\frac{h}{2}\right) - f(x+h) + f(x-h)}{8h\cdot\frac{3}{4}}$$

$$\Rightarrow \boxed{f'(x) = \frac{f(x-h) - 8f\left(x-\frac{h}{2}\right) + 8f\left(x+\frac{h}{2}\right) - f(x+h)}{6h}} \longrightarrow (11)$$

The application of formula (11) depends on the availability of function values at $x \pm \frac{h}{2}$ points. This is the restriction when Richardson extrapolation formula is applied to tabulated function values.

Example : Show that Richardson's extrapolation technique for the given data provides better estimates for first order derivative at $x = 0.5$ for $h = 0.5$, $r = 1/2$, $r = 2$.

x	-0.5	-0.25	0	0.25	0.5	0.75	1.0	1.25	1.5
$f(x) = e^{2x}$	-3.6945	-1.8472	1	1.8472	3.6945	5.5417	7.3890	9.2363	11.0835

Solution : Take $h = 0.5$, $r = \frac{1}{2}$, $x = 0.5$.

Using three-point Central formula,

$$D(h) = \frac{f(x+h) - f(x-h)}{2h}$$

$$\Rightarrow D(0.5) = \frac{f(0.5+0.5) - f(0.5-0.5)}{2 \times 0.5}$$

$$= \frac{f(1) - f(0)}{1} = \frac{7.3890 - 1}{1} = 6.389$$

$$\therefore \boxed{D(0.5) = 6.389} \longrightarrow (1)$$

Now, $D(rh) = \dfrac{f(x+rh) - f(x-rh)}{2(rh)}$

$$\therefore D\left(\frac{1}{2} \times 0.5\right) = \frac{f(0.5+0.25) - f(0.5-0.25)}{2 \times 0.25}$$

$$= \frac{f(0.75) - f(0.25)}{0.5}$$

$$= \frac{5.5417 - 1.8472}{0.5}$$

$$\boxed{D(0.25) = 7.389} \longrightarrow (2)$$

By Richardson extrapolation,

$$f'(x) = \frac{D(rh) - r^2 D(h)}{1 - r^2}$$

$$\Rightarrow f'(0.5) = \frac{D(0.25) - \frac{1}{4} D(0.5)}{1 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{7.389 - \frac{1}{4}(6.389)}{3/4}$$

$$\Rightarrow \boxed{f'(0.5) = 7.7226} \longrightarrow (3)$$

Now taking $h = 0.5$, $x = 0.5$, $r = 2$,

$$D(rh) = D(2\times 0.5) = D(1) = \frac{f(0.5 + 2\times 0.5) - f(0.5 - 2\times 0.5)}{2(2\times 0.5)}$$

$$\Rightarrow D(1) = \frac{f(1.5) - f(-0.5)}{2}$$

$$= \frac{11.0835 - (-3.6945)}{2}$$

$$= 7.3885$$

$$\therefore f'(0.5) = \frac{D(1) - 4D(0.5)}{1-4}$$

$$= \frac{7.3885 - 4(6.389)}{-3} \quad [\text{From } (1)]$$

$$\boxed{f'(0.5) = 6.0558} \longrightarrow (4)$$

- The correct value of $f'(0.5) = 2.e^{2(0.5)}$ is 7.7226.
- Thus from (3) & (4), we observe that estimate of $r = 0.5$ is much better than estimate at $r = 2$.

Practice Questions (9)

(1) By use of Richardson extrapolation, find $f'(1)$ using the approximate formula:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

with $h = 0.4$, 0.2 and 0.1 from the following values:

x	$f(x)$
0.6	0.707178
0.8	0.859892
0.9	0.925863
1.0	0.984007
1.1	1.033743
1.2	1.074575
1.4	1.127986

Reference Books :

* Books mentioned in GE-4 Paper syllabus.
* Higher Engineering Mathematics; B.S. Grewal; Khanna Publishers
* Numerical Methods; G. Pragati & Arvind; Narosa Publishing House
* Advanced Engineering Mathematics; H.C Taneja; dreamtech, Wiley.
* Advanced Engineering Mathematics; Greenberg; Pearson Education.